Abstract

This tutorial on Mathematics explores fundamental calculus concepts, from initial learning hurdles to its historical development, including the Greek "method of exhaustion" and the contributions of Newton and Leibniz. Central to calculus is the limit of a function, a "foundational concept". Understanding limits is paramount, as they form the bedrock for defining derivatives (like $\frac{d}{dx}$) and integrals (like $\int$), enabling the precise formulation of instantaneous change and areas under curves. This chapter illuminates these pivotal ideas through historical context and personal reflection.

Although the foundational concept of limits was presented, as a less-than-exceptional student, these initial ideas remained elusive to me. I watched as my peers navigated differentiation with ease, while the meaning of $\frac{d}{dx}$ stubbornly evaded my grasp. The curriculum then moved onward to integration, the so-called reverse of differentiation, introducing yet another symbolic hurdle: $\int$. This elongated 'S', I learned, represented summation. At that point, calculus felt more like an adversary than an ally. It wasn't until my graduate studies that a few insightful mentors emerged, rescuing me from this intellectual quagmire. They illuminated some of my lingering doubts and subtly redirected my academic trajectory. Years later, upon joining a private engineering college in Kolkata (India) as a lecturer in the Department of Mathematics, a dear colleague gifted me Tom M. Apostol's "Calculus" [2]. To my surprise, the book clearly articulated that the historical development of integration predates that of differentiation. Intriguingly, the pedagogical tradition in Indian schools often begins with the exploration of differentiation, a sequence that stands in contrast to the historical evolution of these mathematical concepts for reasons unknown to me.

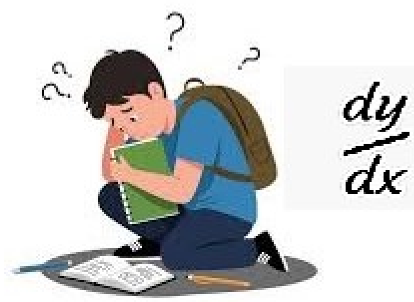

Figure 1: The image captures my abstract interpretation of my battling with calculus (Photo Courtesy: Google).

Historical background:

It all started with the Greeks. Over two millennia ago, the Greeks initiated the quest to determine areas of irregular forms, a pioneering approach known as the *method of exhaustion* [2, 5].

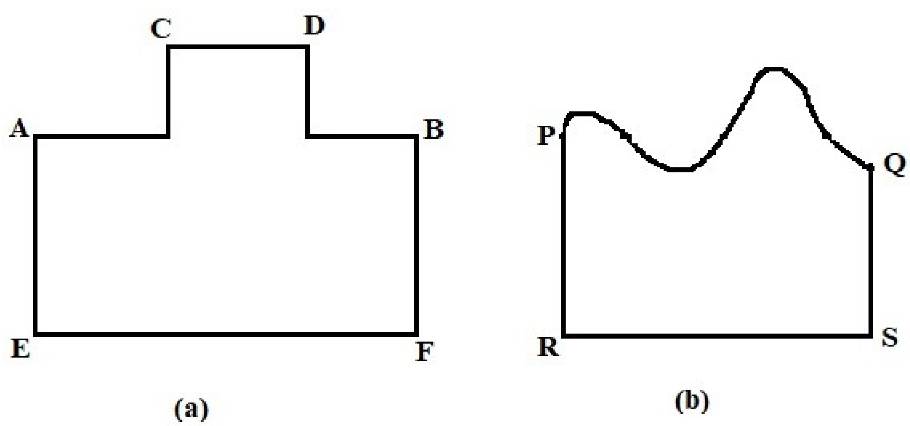

Figure 2: (a) Regular Shape (b) Irregular Shape (Photo Courtesy: Google).

Middle school exercises often introduce area calculation through regular shapes (e.g., Figure 2a, where students can systematically decompose the figure into manageable rectangles and compute the area using basic multiplication. That is, to determine the area of a given region, we can iteratively inscribe polygonal regions that progressively approximate the target. Starting with a simple polygon whose area is readily calculable, we refine the approximation by inscribing polygons with an increasing number of sides, aiming to exhaust the original region. This straightforward approach, however, reveals its limitations when confronted with irregular shapes like Figure 2b, for which the construction of auxiliary lines to form complete rectangles fails to encompass the entire area. Archimedes (287-212 BCE) masterfully derived exact formulas for the area of a circle and other specific geometric figures. However, further advancement of his method of exhaustion remained largely dormant for nearly eighteen centuries. This hiatus persisted until the advent of algebraic notation and techniques became integral to mathematics. The elementary algebra familiar to modern high school students was entirely absent in Archimedes' era, rendering the extension of his ingenious method to broader classes of regions exceedingly challenging without a concise and simplified means of expressing complex calculations. A subtle yet transformative evolution in mathematical notation commenced in the 16th Century CE. The unwieldy Roman numeral system gradually yielded to the now-ubiquitous Hindu-Arabic numerals, while the nascent symbols $+$ and $-$ emerged, and the inherent advantages of decimal notation began to gain recognition. Concurrently, the remarkable achievements of Italian mathematicians Tartaglia, Cardano, and Ferrari in solving cubic and quartic equations algebraically ignited a surge of mathematical activity, fostering the growth and acceptance of a refined and potent algebraic idiom. The widespread adoption of judiciously chosen algebraic symbols breathed new life into the ancient method of exhaustion, leading to a wealth of initial discoveries in the 16th Century by luminaries such as Cavalieri, Torricelli, Roberval, Fermat, Pascal, and Wallis [1].

As the eighteenth century dawned, a clash of titans loomed: Gottfried Wilhelm Leibniz (1646-1716), the German polymath, and Sir Isaac Newton (1642-1726), the English luminary. For over a decade, these brilliant minds engaged in a protracted and acrimonious public dispute, each fiercely asserting his claim to the intellectual genesis of calculus—that pivotal branch of mathematical analysis that unlocks the secrets of everything from geometric forms to the celestial dance of planets. At the zenith of the fierce calculus controversy, the towering intellects of Newton and Leibniz engaged in a spirited, often clandestine, battle. Through the shadowy avenues of anonymously penned treatises and publications bearing another's name, they cast their barbs. Each, a luminary recognized across Europe, leveraged their formidable reputation to sway opinion. Loyal colleagues rallied to their banners, cleaving the intellectual landscape into two distinct factions. They amassed weighty compendiums of evidence, penned voluminous defenses, and bristled with indignation at each perceived slight from their rival. Had fate not intervened with Leibniz's passing in 1716, this intellectual duel would undoubtedly have persisted, a testament to their unwavering conviction.

Figure 3: (a) Sir Issac Newton (b) Gottfried Leibniz (Photo Courtesy: Google).

The question of rightful primacy in the discovery of calculus between Newton and Leibniz remains a nuanced and occasionally contentious subject within the scientific community. However, within the educational sphere of high school curricula, both figures are rightfully accorded respect for their independent and profound contributions to this foundational field. As a student, the historical controversy surrounding Newton and Leibniz's independent development of calculus remained beyond my awareness; it was only in my later years, while engaged in teaching, that these fascinating details came to light [3].

Echoes of a Mathematical Divide:

Mathematics, in its essence, is an elegant abstraction drawn from the physical world we perceive. Ideally, this fundamental discipline should exist as a unified whole. However, the academic landscape in India often reveals a distinct division between Pure Mathematics, celebrated for its abstract beauty, and Applied Mathematics, valued for its real-world applicability, sometimes fostering an unnecessary tension between these perspectives. Indeed, in my earlier years, I engaged in spirited debates with a close colleague, navigating this very divide. My colleague, a pure mathematician through and through, had a singular starting point for any mathematical inquiry I posed: the concept of a function. This foundational approach, while thorough, occasionally tested my patience. One day, my frustration bubbling over, I implored him for a direct answer to my specific question. His response, delivered with unwavering conviction, resonated with the core of his mathematical world-view: "*Everything is function. Without grasping this fundamental truth, nothing else will truly make sense*". The root of my calculus anxiety lay in a discomfort with its foundational pillar: the concept of limits, which itself rests upon the bedrock of the function. Understanding the definition of a function presupposes a grasp of set theory, for which I would direct the reader to Paul Richard Halmos' renowned treatise *Naive Set Theory* [8].

Paul R. Halmos (The Nascent Scholar):

From the heart of Budapest, a young mind named P*á*l, later to be known to the world as Paul Richard Halmos (1916-2006), em-

Figure 4: My friend in discussion with my other colleagues (Animated by: Author himself).

barked on a journey marked by both early sorrow and extraordinary intellectual ascent. Even in infancy, fate cast a shadow with the passing of his mother, Paula, a mere six months after his arrival. His father, a physician of keen foresight, sensed the gathering storm clouds over Europe, a subtle premonition of the continent's impending turmoil. Halmos's formative years unfolded in the academic cloisters of Budapest until the age of thirteen, whereupon the trajectory of his life shifted across the Atlantic to the bustling metropolis of Chicago. Here, in a remarkable twist of circumstance, four years of formal schooling were bypassed, a testament perhaps to an innate brilliance that transcended conventional educational pathways. The transition was not without its initial hurdles; the young Halmos arrived on American shores possessing a linguistic repertoire of Hungarian and German, yet devoid of English. His first day in a new scholastic environment became a charming vignette of intellectual resourcefulness, navigating his way through a few shared Latin and French phrases with a guiding teacher. Within half a year, the English language yielded to his sharp intellect, allowing him to converse in a "rapid, incorrect, ungrammatical, colloquial" manner, a testament to his swift adaptation. While still fifteen years old he entered the University of Illinois to study chemical engineering. He had considered other options such as studying law at a law school but opted for chemistry. A year later, the allure of chemistry waned, the tactile nature of the discipline proving less captivating than the abstract elegance he sought. This led him to explore the realms of mathematics and philosophy, though his initial engagement with mathematics did not ignite any particular brilliance [6]. The aca-

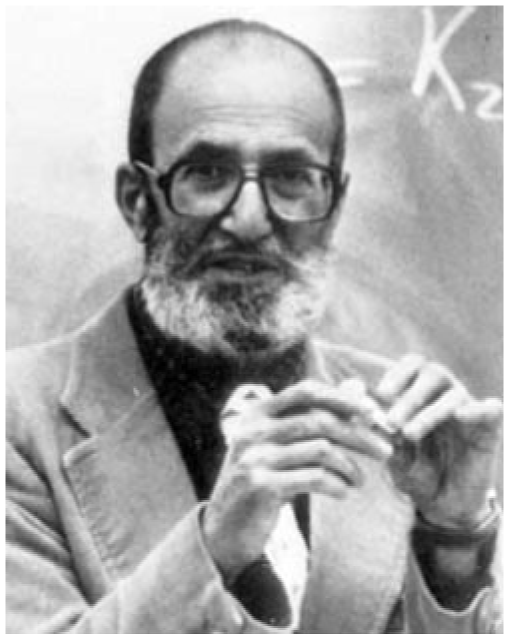

Figure 5: Paul Richard Halmos (Photo Courtesy: Google).

demic year of $1935-36$ proved to be a pivotal juncture. It was the unexpected stumble in his *Philosophy Master's oral examination* that served as a catalyst, redirecting his path towards the mathematical landscapes that would ultimately define his legacy. September of that year marked another significant milestone: his inaugural foray into the world of pedagogy, teaching the foundational principles of freshman algebra. Even prior to this decisive shift, at the tender age of fifteen, the nascent scholar had already matriculated at the University of Illinois, initially with the intention of pursuing chemical engineering. The paths of law and chemistry had been considered, the latter ultimately chosen, yet destiny, it seemed, had a different, more abstract, and profoundly influential course charted for the Hungarian émigré who would become a titan in the domains of mathematical logic, probability theory, operator theory, ergodic theory, and the intricate geometry of functional analysis, particularly within the elegant structures of Hilbert spaces. His life story, therefore, is not merely a recitation of academic achievements, but a compelling narrative of resilience, intellectual curiosity, and an eventual, triumphant embrace of the abstract world that would forever bear the imprint of his profound contributions.

Concept of function:

No concept in Mathematics, especially in Calculus, is more fundamental than the concept of a *function*, the term which was first initiated by the German mathematician Gottfried Wilhelm Leibniz, in his 1633 letter to Sir Issac Newton (the tug of war between these two stalwarts have already been discussed in the previous section). To define function we need to understand the following: *If A and B are two non-empty sets*[1]*, the cartesian product $A \times B$ of A and B is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$. That is,*

$$A \times B = \{(a, b) \mid a \in A, b \in B\}. \tag{1}$$

René Descrates' cartesian plane is a well set example; where one can consider the $\mathbb{X}-$ axis and $\mathbb{Y}-$ axis as two sets of infinitely many points and whose cartesian product denoted by $\mathbb{X} \times \mathbb{Y}$ will generate the entire cartesian plane with four quadrants. Now let A and B be two sets. Then, *a function from A to B is a set f of ordered pairs in $A \times B$ such that for each $x \in A$ there exists a unique $y \in B$ with $(x, y) \in f$. The notation $f : A \to B$ is often used to indicate that f is a function from A to B or mapping of A into B*

[1] A set is a well-defined collection of distinct objects.

or that f maps A into B. It is customary to write

$$y = f(x), \tag{2}$$

while we sometimes refer to y as the value of f at x. The set A of first elements of a function f is called the *domain of f*, denoted by $\mathcal{D}(f)$, whereas the set of all second elements in f is called the *range of f*, denoted by $\mathcal{R}(f)$. In the Figure 6 if we consider the

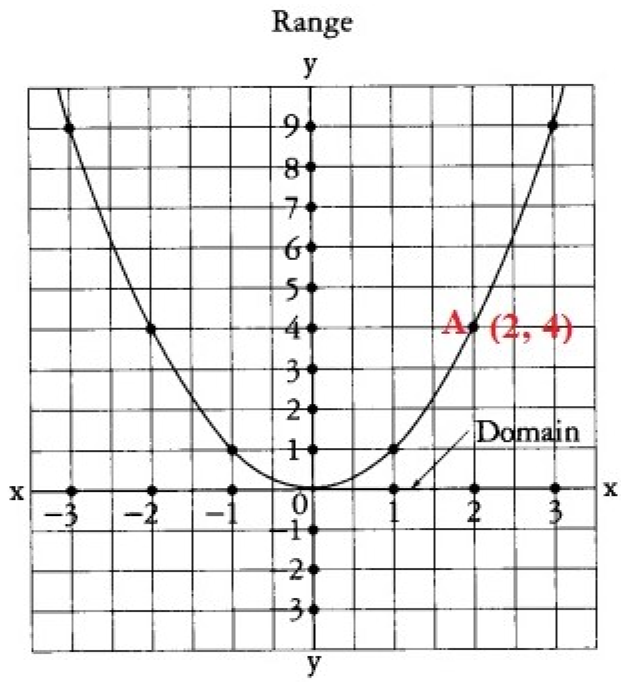

Figure 6: Cartesian Plane (Photo Courtesy: Google).

point $A(2, 4)$, then by the definition of function we can say that 4 is the image of 2. It is also to be noted from the definition of a function that although $\mathcal{D}(f) = A$, one only has $\mathcal{R}(f) \subset B$. An example will clarify this point. Consider the function $g : \mathbf{R} \to \mathbf{R}$, where $\mathbf{R}$ is the set of real numbers and the function is given by

$$y = g(x) = \sin(x). \tag{3}$$

The reader can easily check that as $-1 \leq \sin(x) \leq 1$, so $\mathcal{D}(g) = \mathbf{R}$ while $\mathcal{R}(g) = [-1,\ 1] \subset \mathbf{R}$. Geometrically also one can understand this from the following figure[2]. One can also imagine a mathe-

[2]Introduction to Real Analysis by Bartle and Sherbert.

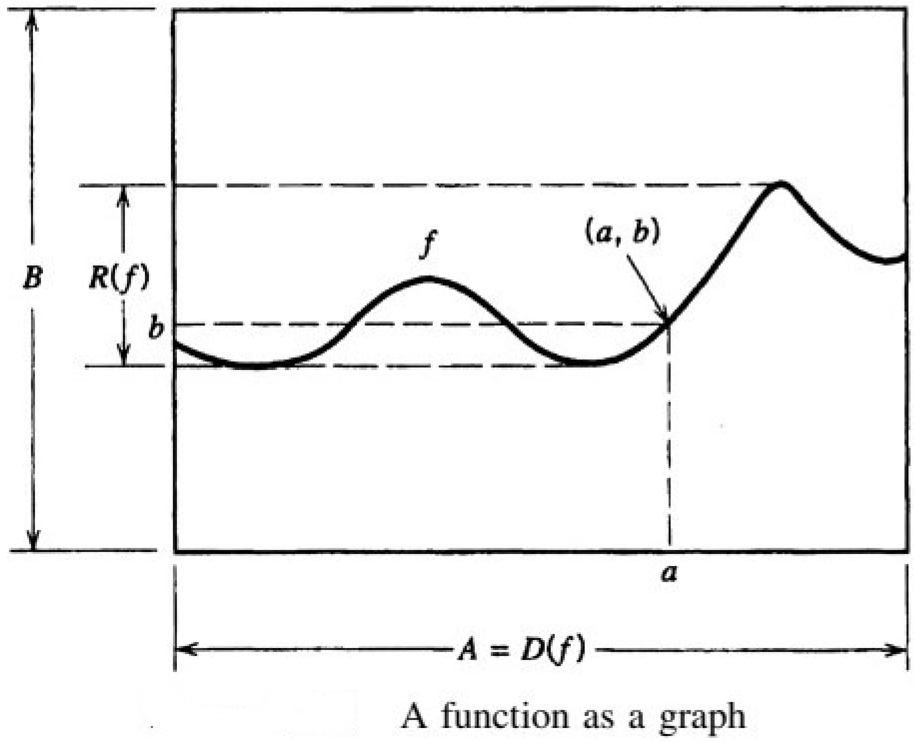

Figure 7: Schematic of a function with domain and range clearly shown (Image Courtesy: Bartle and Sherbert).

matical function as an elegant machine which is fed with an input such as a raw ingredient. Then through a precise sequence of operations, the machine transforms that input into a specific output i.e. the finished product. This output is uniquely determined by the input and the machine's internal workings, mirroring the deterministic nature of mathematical relationships. A concrete physical example can be a simple pulley system. If one applies a certain force (the input) to one end of the rope, the pulley (the function) will output a corresponding lifting force (the output) on the object attached to the other end, governed by the mechanics of the system. Another good example would be to consider: a car's speedometer. The input is the rotational speed of the car's wheels, and the speedometer mechanism (the function) transforms this into a corresponding output: the car's speed displayed in miles per hour or kilometres per hour. For intricate details of functions, readers are referred the book written by Robert G. Bartle and Donald R. Sherbert [4].

Descartes:

René Descartes ($1596\breve{}1650$), a mind of the first order, profoundly shaped the intellectual landscape as a creative mathematician, an insightful scientific thinker, and an original metaphysician. His intellectual journey unfolded with mathematics as his initial focus, followed by his exploration of natural science (or "natural philosophy"), culminating in his metaphysical inquiries. Legend has it that the genesis of the Cartesian plane, René Descartes' ingenious method for representing points through numerical coordinates, arose from a seemingly mundane observation. As the tale unfolds, Descartes, watching a fly traverse his bedroom ceiling, conceived of a system to precisely define its location. He recognized that the fly's position could be uniquely determined by its distances from the room's corners, an insight that elegantly translated into the concept of reference axes – the very foundation of the x and y coordinates. This moment of contemplative observation, whether factual or apocryphal, underscores the profound simplicity at the heart of Descartes' innovation, a unifying framework that seamlessly intertwined the realms of algebra and geometry[7].

More on functions:

The functions mainly are of two types: (a) one-one function and (b) onto function. Let $f : A \to B$ be a function from A to B.

- *The function f is said to be injective (or to be one-one* if whenever $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.
- The function f is said to be *surjective* or to map A onto B if $f(A) = B$; i.e., if the range $\mathcal{R}(f) = B$.
- If a function is both injective and surjective. such a function is known as *bijective* function.

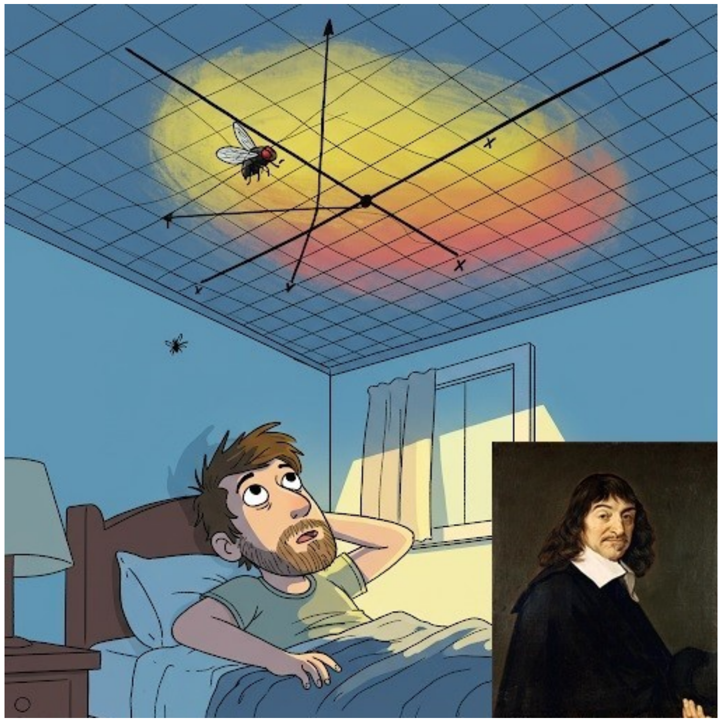

Figure 8: An AI generated image of Descartes' vision of coordinate system (Image Courtesy: Google).

The definition of function clearly hints that an element of the domain A can't have two different images. When you stand in front of the mirror, do you expect the mirror to reflect two different images of yours? No. In this case, the real world to which you belong is the set A, the world behind the mirror is set B and the mirror is the function f. You anticipate seeing a single reflection of yourself, and naturally, if someone stands beside you, their reflection would appear alongside yours. So a mirror is always a one-one function. Now to every reflection appeared on the virtual side of the mirror has a pre-image in the real side. Hence by definition, one can consider a mirror to be both one-one and onto function and hence a bijective one. Now let us take another example. Are you familiar with the physics demonstration where a magnifying glass focuses sunlight to a single, intense point, capable of igniting paper placed beneath? Could we consider the magnifying glass as analogous to an onto function, where the scorched point on the paper represents the range, and each of the converging photons of light acts as a pre-image mapping to that single point

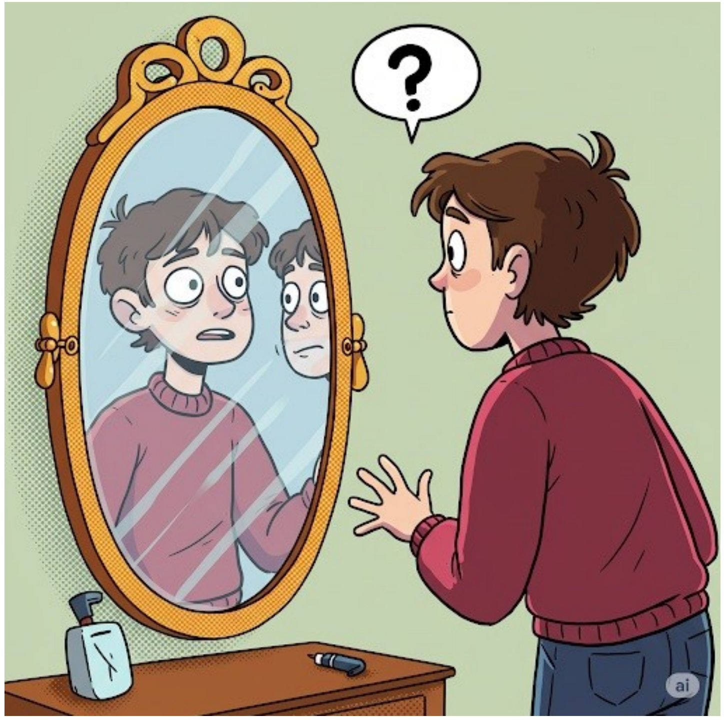

Figure 9: An AI generated image of a mirror which is not a function (Image Courtesy: Google).

of combustion (where the set of the light photons is the domain set)? Pedagogically, yes.

Figure 10: An AI generated image of a magnifying glass which is an onto function (Image Courtesy: Google).

Thought of the day:

Let me pause here, leaving the readers with a contemplative reflection: Let $A = B = \{x \in \mathbf{R} : -1 \le x \le 1\}$ and consider the subset $C = \{(x, y) : x^2 + y^2 = 1$ of $A \times B$. Is this set a function? Explain!

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